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A SIMPLIFIED STABILITY THEORY OF GENERAL SHELLS

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Abstract—The equilibrium equations of a shell are first expressed in a set of orthogonal frames the orientation of which changes along the coordinate curves, taken as the principal curvatures. Then using the Euler approach and considering adjacent equilibrium states near the critical state, a set of compact vector equations are derived for the stability of general shells. The effects of shear deformations, local rotations, and motion-dependent forces are conveniently taken into account in this approach. Pertinent equations for shells of specific geometry are easily derived from the general equations presented here. By way of illustration, the governing equations for stability of cylindrical shells are obtained from the general equations. These equations in their general form, and in their special form for plates, contain the effects of local rotations as well as the other types of classical shell deformations. They can therefore, be used for the in-plane instability analysis of composite plates, shells and/or gridworks and also for the analysis of *membranal buckling* of shells. By way of illustration, this new feature is utilized in the development of a continuum model for gridwork instability.

1. INTRODUCTION

The literature on the theory of shells is very extensive and, aside from numerous papers, there exist a number of excellent texts on the subject of thin shells (Koiter, 1960; Niordson, 1969; Donnel, 1976; Green and Zerna, 1968). With regard to stability investigations, a good deal of the literature is concerned with shells of specific geometry. This is to be expected given the importance of such shell geometries as cylindrical, spherical, and general shells of revolution in numerous applications. Attempts have also been made towards the construction of general stability equations for shells, in component form. Useful as these derivations are, they do not lend themselves to compact notation for ease of manipulations and they impede a holistic understanding of shell behavior. There are, also, some papers which adopt a rather general approach and consider the shell as a Cosserat surface (Naghdi, 1972; Green and Naghdi, 1971). While of considerable theoretical interest these approaches are not normally used for engineering applications. Accordingly there exists a need to derive a set of compact and yet physically plausible equations for stability of general shells of arbitrary geometry from which equations for specific geometries may be derived. From a theoretical point of view, such an approach would bring to light the commonality in the stability theory of shells and would therefore serve as a unifying foundation.

To obtain general results which are at the same time useful in applications it is necessary to strike a reasonable balance between the details of generality and the simplicity of expressions derived. In the following, we outline a technically plausible stability theory of general shells and by using concepts based on dynamic–kinematic analogy between spatial rods and relative motion in dynamics, we express the pertinent equations of our theory in a compact vector notation. A similar approach has been adopted by the authors to establish a technical stability theory of spatial rods under general motion-dependent loading (Farshad and Tabarrok, 1987).

As one of the outcomes of the present theory, we have derived the stability equations of cylindrical shells, embodying the effects of local rotations and also the stability equations of Cosserat-type plates. These equations may be conveniently applied to the stability analysis of composite shells and plates, and also gridworks undergoing in-plane, or a

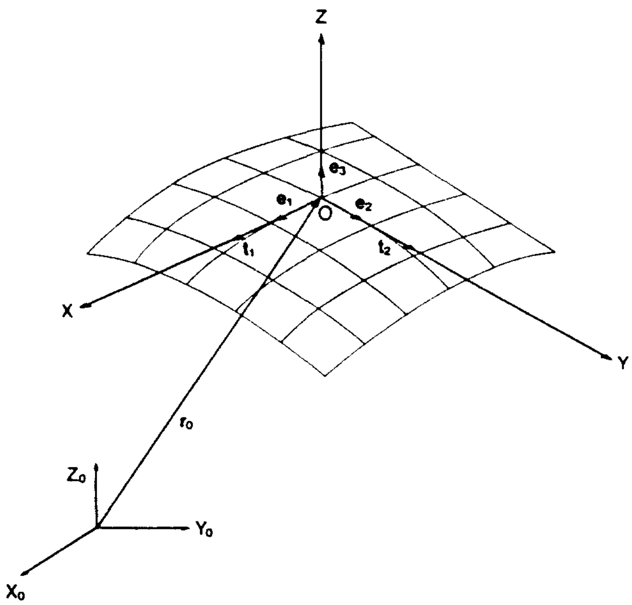


Fig. 1. Geometry of the curved surface.

membranal model of buckling. An illustrative example of such a potential application is also worked out.

2. GEOMETRICAL PRELIMINARIES

Consider an arbitrary surface as shown in Fig. 1. We may locate a point on this surface by the position vector $\mathbf{r}_o$ which may be considered as a function of coordinate parameters θ_1,θ_2. Thus

$$\mathbf{r}_o = \mathbf{r}_o(\theta_1, \theta_2). \tag{1}$$

Next, let us introduce a local orthogonal coordinate system X,Y,Z attached to the shell middle surface in such a manner that X,Y axes lie along the lines of principal curvature of this surface and the Z axis lies normal to it. Then, the tangent vectors associated with the X,Y axes may be expressed as

$$\mathbf{t}_\alpha = \frac{\partial \mathbf{r}_o}{\partial \theta_\alpha}, \quad \alpha = 1,2. \tag{2}$$

Now, defining the scale factors (Lamé parameters) A_1,A_2 as

$$A_\alpha = |\mathbf{t}_\alpha|, \quad \alpha = 1,2 \tag{3}$$

the unit base vectors, $\mathbf{e}_\alpha$, may be expressed as

$$\mathbf{e}_\alpha = \frac{\mathbf{t}_\alpha}{A_\alpha}, \quad \alpha = 1,2 \text{ (no sum on } \alpha). \tag{4}$$

The unit normal vector, $\mathbf{e}_3$, is thus determined as

$$\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2. \tag{5}$$

Let us now consider an element of the shell bounded by two pairs of infinitesimally adjacent

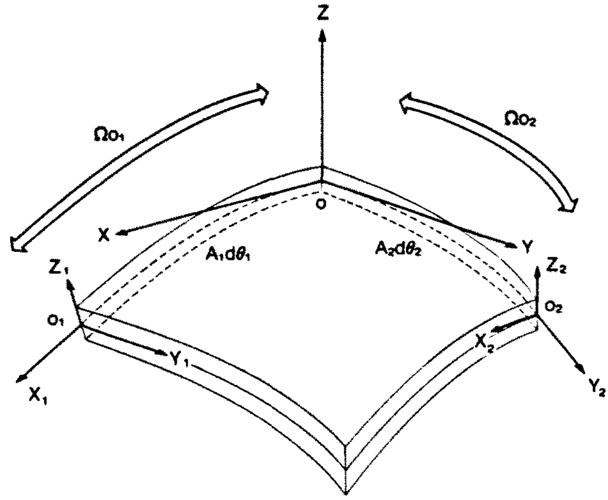


Fig. 2. An element of the shell surface.

lines of curvature having the initial lengths $A_1 d\theta_1$ and $A_2 d\theta_2$ (see Fig. 2). In addition to the previously defined X,Y,Z axes, at point O, we now introduce two new sets of axes $X_\alpha, Y_\alpha, Z_\alpha$ ($\alpha = 1,2$) attached to the neighboring points O_1, O_2, located at distances $A_\alpha d\theta_\alpha$ ($\alpha = 1,2$) from the local original. These new axes too are oriented along the lines of principle curvature at points O_1 and O_2.

In the general case, the orientations of $X_\alpha, Y_\alpha, Z_\alpha$ would differ from those of the X,Y,Z axes. However, through some rotation matrices it is possible to relate these systems. For the case of infinitesimal differences in the orientations of these axes, the rotation matrices take particularly simple forms and we may express the *rate* of charge of orientation of $X_\alpha, Y_\alpha, Z_\alpha$ ($\alpha = 1,2$) with respect to X,Y,Z by some skew symmetric matrices. For this case, it is convenient to represent the transformation via the skew symmetric matrices by cross product terms of orientation vectors $\mathbf{\Omega}_{o_\alpha}$ ($\alpha = 1,2$). The six components of Ω_{o_1} and Ω_{o_2} will not be independent. Indeed, if we designate the independent components by a, b, c and d we may express the general form of the vectors $\mathbf{\Omega}_{o_\alpha}$ as follows (see for instance Love)

$$\begin{aligned} \mathbf{\Omega}_{o_1} &= \Omega_{o_{11}}\mathbf{e}_1 + \Omega_{o_{12}}\mathbf{e}_2 + \Omega_{o_{13}}\mathbf{e}_3 \\ &\equiv (o)\mathbf{e}_1 + (-a)\mathbf{e}_2 + (c)\mathbf{e}_3 \end{aligned} \tag{6}$$

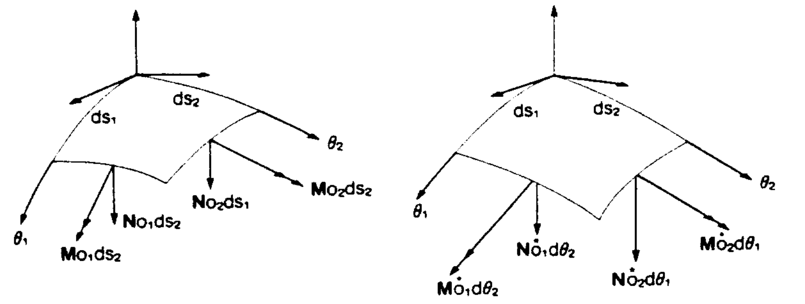
$$\begin{aligned} \mathbf{\Omega}_{o_2} &= \Omega_{o_{21}}\mathbf{e}_1 + \Omega_{o_{22}}\mathbf{e}_2 + \Omega_{o_{23}}\mathbf{e}_3 \\ &\equiv (b)\mathbf{e}_1 + (o)\mathbf{e}_2 + (-d)\mathbf{e}_3. \end{aligned} \tag{7}$$

The quantities a, b, c, d will, in general, be functions of θ_α. These quantities are related to the Lamé parameters as follows:

$$c = -\frac{1}{A_2}\frac{\partial A_1}{\partial \theta_2}, \quad d = -\frac{1}{A_1}\frac{\partial A_2}{\partial \theta_1}$$
$$\frac{\partial a}{\partial \theta_2} = -bc, \quad \frac{\partial b}{\partial \theta_1} = -ad, \quad \frac{\partial c}{\partial \theta_2} + \frac{\partial d}{\partial \theta_1} = ab. \tag{8}$$

Having defined the $X_\alpha, Y_\alpha, Z_\alpha$ ($\alpha = 1,2$) systems, and their relationships with X,Y,Z, we wish to relate the derivatives of a vector quantity, in the $X_\alpha Y_\alpha Z_\alpha$ systems, to its derivatives in the X,Y,Z system. This can be done in analogy with the kinematics of motion in the moving (rotating) coordinate system. Thus, representing a vector quantity by $(\_)$ we have that

$$\left.\frac{\partial(\_)}{\partial \theta_\beta}\right|_{XYZ} = \left.\frac{\partial(\_)}{\partial \theta_\beta}\right|_{X_\alpha Y_\alpha Z_\alpha} + \mathbf{\Omega}_{o_\alpha} \times (\_)$$



(a) (b)

Fig. 3. Internal force and moment resultants: (a) per unit length; (b) per unit value of coordinates.

or symbolically

$$\frac{\partial(\ )}{\partial\theta_\beta} = (\ )_{,\beta}+\Omega_{o_\alpha}\times(\ ), \quad \alpha = 1,2, \quad \beta = 1,2 \tag{9}$$

where $(\ )_{,\beta}$ implies a derivative in the local $(X_\alpha Y_\alpha Z_\alpha)$ coordinate system.

3. PREBUCKLING STATE OF THE SHELL

Consider now a shell of arbitrary geometry loaded in some manner. Let the internal force resultants at any point of this shell be denoted by $\mathbf{N}_{o_\alpha}$ and $\mathbf{M}_{o_\alpha}$ ($\alpha = 1,2$). The former represents the force quantities acting on sectional planes normal to the θ_α directions while the latter denotes the moment quantities associated with these sections.

The subscript "o" associates these vector quantities with the prebuckling configuration of the shell. The internal force resultants are defined per unit deformed *length* of the shell. Let the corresponding quantities, defined per unit value of curvilinear coordinates θ_α ($\alpha = 1,2$), be denoted by $\mathbf{N}^*_{o_\alpha}$ and $\mathbf{M}^*_{o_\alpha}$, respectively, so that

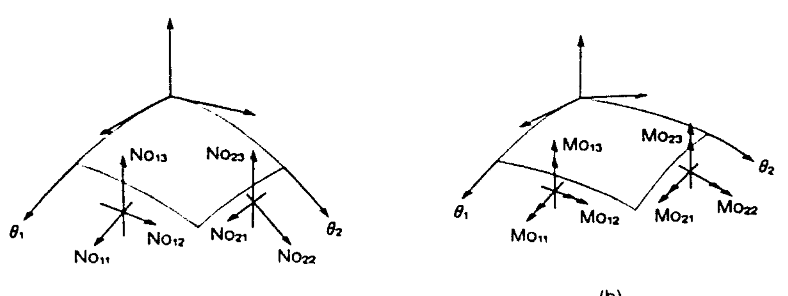
$$\mathbf{N}_{o_\alpha}\,\mathrm{d}s_\beta = \mathbf{N}^*_{o_\alpha}\,\mathrm{d}\theta_\beta, \quad \mathbf{M}_{o_\alpha}\,\mathrm{d}s_\beta = \mathbf{M}^*_{o_\alpha}\,\mathrm{d}\theta_\beta, \quad \alpha = 1,2, \quad \beta = 1,2. \tag{10}$$

Now, noting that $\mathrm{d}s_\beta = A_\beta\,\mathrm{d}\theta_\beta$, we may express the above relations as follows:

$$\mathbf{N}_{o_\alpha} = \frac{A_\alpha}{G}\mathbf{N}^*_{o_\alpha}, \quad \mathbf{M}_{o_\alpha} = \frac{A_\alpha}{G}\mathbf{M}^*_{o_\alpha} \tag{11}$$

where $G = A_1A_2$.

The internal force resultants $(\mathbf{N}_{o_\alpha}, \mathbf{M}_{o_\alpha})$ and their corresponding quantities ($\mathbf{N}^*_{o_\alpha}$ and $\mathbf{M}^*_{o_\alpha}$), are defined in Figs 3(a) and (b), respectively. Figures 4(a) and (b) depict the internal resultant force field $(\mathbf{N}_{o_\alpha}, \mathbf{M}_{o_\alpha})$ in component form.



(a) (b)

Fig. 4. Components of internal force and moment resultants.

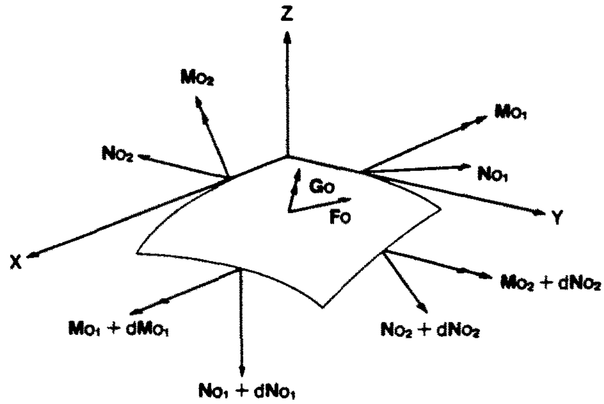


Fig. 5. Free body diagram for an element of the shell.

We may express the internal stress resultant vectors $(\mathbf{N}_{o_\alpha}, \mathbf{M}_{o_\alpha})$ and $(\mathbf{N}^*_{o_\alpha}, \mathbf{M}^*_{o_\alpha})$ in terms of their components in the XYZ system as follows:

$$\mathbf{N}_{o_\alpha} = N_{o_{\alpha i}}\mathbf{e}_i, \quad \mathbf{M}_{o_\alpha} = M_{o_{\alpha i}}\mathbf{e}_i \quad (\text{sum on } i = 1, 2, 3) \tag{12}$$

also

$$\mathbf{N}^*_{o_\alpha} = N^*_{o_{\alpha i}}\mathbf{e}_i, \quad \mathbf{M}^*_{o_\alpha} = M^*_{o_{\alpha i}}\mathbf{e}_i. \tag{13}$$

Now, referring to the free body diagram of the shell element, in Fig. 5, subjected to applied force vector $\mathbf{F}_o$ and moment vector $\mathbf{G}_o$, we may write the equations of equilibrium of the element as follows:

$$\frac{\partial \mathbf{N}^*_{o_\alpha}}{\partial \theta_\alpha} + \mathbf{F}_o = \mathbf{0} \tag{14}$$

$$\frac{\partial \mathbf{M}^*_{o_\alpha}}{\partial \theta_\alpha} + \mathbf{t}_\alpha \times \mathbf{N}^*_{o_\alpha} + \mathbf{G}_o = \mathbf{0} \tag{15}$$

or, utilizing the relative differentiation relations given in eqn (9), we may express the equilibrium equations as

$$(\mathbf{N}^*_{o_\alpha})_{,\alpha} + \mathbf{\Omega}_{o_\alpha} \times \mathbf{N}^*_{o_\alpha} + \mathbf{F}_o = \mathbf{0} \tag{16}$$

$$(\mathbf{M}^*_{o_\alpha})_{,\alpha} + \mathbf{\Omega}_{o_\alpha} \times \mathbf{M}^*_{o_\alpha} + \mathbf{t}_\alpha \times \mathbf{N}^*_{o_\alpha} + \mathbf{G}_o = \mathbf{0}. \tag{17}$$

4. EQUILIBRATING STRESS FIELDS

The two vectorial equilibrium equations, expressed in terms of the four internal stress vector resultants, are an indication of the statical indeterminacy of the shell problem. As such, it is possible to establish an infinite number of equilibrating fields. This is best done by the introduction of some stress functions. The vectorial form of the equilibrium equations suggests the following form of stress functions:

$$\mathbf{N}^*_{o_1} = -\frac{\partial \mathbf{L}}{\partial \theta_2}, \quad \mathbf{N}^*_{o_2} = \frac{\partial \mathbf{L}}{\partial \theta_1} \tag{18}$$

$$\mathbf{M}^*_{o_1} = \frac{\partial \mathbf{K}}{\partial \theta_2} - \mathbf{t}_2 \times \mathbf{L}, \quad \mathbf{M}^*_{o_2} = -\frac{\partial \mathbf{K}}{\partial \theta_1} + \mathbf{t}_1 \times \mathbf{L}. \tag{19}$$

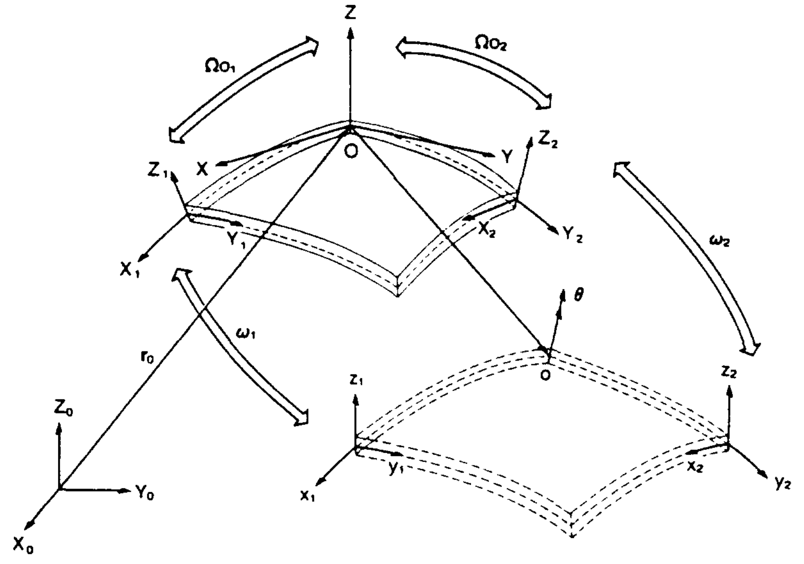


Fig. 6. A shell element in its initial and perturbed states.

It is a simple matter to show that the stress resultants defined in terms of the vector stress functions, **L** and **M**, as shown above, provide the homogeneous solution to the equilibrium equations, eqns (16) and (17). The particular solution can be easily incorporated in the stress function expressions if the force vectors $\mathbf{F}_0$ and $\mathbf{G}_0$ can be derived from some potential functions. The definitions in eqns (18) and (19) can be written more compactly by using the two-dimensional alternating tensor $e_{\alpha\beta}$. Then we may write

$$\mathbf{N}^*_{o_\alpha} = e_{\alpha\beta}\frac{\partial \mathbf{L}}{\partial \theta_\beta} \tag{20}$$

$$\mathbf{M}^*_{o_\alpha} = e_{\alpha\beta}\left[\frac{\partial \mathbf{K}}{\partial \theta_\beta} + \mathbf{t}_\beta \times \mathbf{L}\right]. \tag{21}$$

For a general Cosserat-type surface, having non-zero $\mathbf{e}_3 M_{o_{\alpha}3}$ components, the six components of stress function vectors, **L** and **K**, are independent. In the classical shell theory, however, the twisting moment, normal to the shell surface, is suppressed. In that case, some relations exist amongst the components of **L** and **K** (see, e.g. Goldenveiser (1961)).

5. STABILITY EQUATIONS

Consider now an adjacent equilibrium configuration of the shell as compared with the previously identified initial state. This perturbed configuration is characterized by two deformation vectors **u** and $\boldsymbol{\phi}$. Physically, **u** signifies the displacement of the origin of the XYZ system and $\boldsymbol{\phi}$ identifies the rotation of the normal vector, relative to its initial orientation. Figure 6 depicts the shell element in the initial and the perturbed states.

Now, let the reference frames along the edges of the element, in the perturbed configuration be $x_\alpha y_\alpha z_\alpha$ ($\alpha = 1,2$) as shown in Fig. 6. Further, let the vector rate of change of orientation of $x_\alpha y_\alpha z_\alpha$ systems relative to $X_\alpha Y_\alpha Z_\alpha$ be denoted by ω_α ($\alpha = 1,2$). To derive the stability equations of the shell we assume that the initial configuration with internal stress resultants $\mathbf{N}^*_{o_\alpha}$, $\mathbf{M}^*_{o_\alpha}$, is the critical state. We further assume that the perturbed state is another possible equilibrium configuration. Now, the geometrical and the physical quantities of the shell in the perturbed state are related to their initial counterparts by the following perturbation relations:

$$\mathbf{r} = \mathbf{r}_o + \mathbf{u}$$
$$\mathbf{N}^*_\alpha = \mathbf{N}^*_{o_\alpha} + \mathbf{n}^*_\alpha$$
$$\mathbf{M}^*_\alpha = \mathbf{M}^*_{o_\alpha} + \mathbf{m}^*_\alpha$$
$$\mathbf{\Omega}_\alpha = \mathbf{\Omega}_{o_\alpha} + \boldsymbol{\omega}_\alpha$$
$$\mathbf{F} = \mathbf{F}_o + \mathbf{f}$$
$$\mathbf{G} = \mathbf{G}_o + \mathbf{g}. \tag{22}$$

5.1. *Perturbed equilibrium equations*

In the perturbed state, the equilibrium equations of the shell, as referred to the initial state, take the form

$$\mathbf{N}^*_{\alpha,\alpha} + \mathbf{\Omega}_\alpha \times \mathbf{N}^*_\alpha + \mathbf{F} = \mathbf{0} \tag{23}$$

$$\mathbf{M}^*_{\alpha,\alpha} + \mathbf{\Omega}_\alpha \times \mathbf{M}^*_\alpha + \mathbf{t}_\alpha \times \mathbf{N}^*_\alpha + \mathbf{G} = \mathbf{0}. \tag{24}$$

By using eqns (22) we may express these equations as

$$\mathbf{N}^*_{o_{\alpha,\alpha}} + \mathbf{n}^*_{\alpha,\alpha} + (\mathbf{\Omega}_{o_\alpha} + \boldsymbol{\omega}_\alpha) \times (\mathbf{N}^*_{o_\alpha} + n^*_\alpha) + \mathbf{F}_o + \mathbf{f} = \mathbf{0} \tag{25}$$

$$\mathbf{M}^*_{o_{\alpha,\alpha}} + \mathbf{m}^*_{\alpha,\alpha} + (\mathbf{\Omega}_{o_\alpha} + \boldsymbol{\omega}_\alpha) \times (\mathbf{M}^*_{o_\alpha} + \mathbf{m}^*_\alpha) + \mathbf{t}_\alpha \times (\mathbf{N}^*_{o_\alpha} + \mathbf{n}^*_\alpha) + \mathbf{G}_o + \mathbf{g} = \mathbf{0}. \tag{26}$$

Now, subtracting eqns (16) and (17) from eqns (25) and (26), and neglecting the second-order non-linear terms, we find the equations for the perturbations as follows:

$$\mathbf{n}^*_{\alpha,\alpha} + \mathbf{\Omega}_{o_\alpha} \times \mathbf{n}^*_\alpha + \boldsymbol{\omega}_\alpha \times \mathbf{N}^*_{o_\alpha} + \mathbf{f} = \mathbf{0} \tag{27}$$

$$\mathbf{m}^*_{\alpha,\alpha} + \mathbf{\Omega}_{o_\alpha} \times \mathbf{m}^*_\alpha + \boldsymbol{\omega}_\alpha \times \mathbf{M}^*_{o_\alpha} + \mathbf{t}_\alpha \times \mathbf{n}^*_\alpha + \mathbf{g} = \mathbf{0}. \tag{28}$$

In terms of force and moment per unit length quantities, namely $\mathbf{n}_\alpha, \mathbf{m}_\alpha$, eqns (27) and (28) may be expressed as

$$(A_2\mathbf{n}_1)_{,1} + (A_1\mathbf{n}_2)_{,2} + \mathbf{\Omega}_{o_1} \times (A_2\mathbf{n}_1) + \mathbf{\Omega}_{o_2} \times (A_1\mathbf{n}_2) + \boldsymbol{\omega}_1 \times (A_2\mathbf{N}_{o_1}) + \boldsymbol{\omega}_2 \times (A_1\mathbf{N}_{o_2}) + \mathbf{f} = \mathbf{0} \tag{29}$$

$$(A_2\mathbf{m}_1)_{,1} + (A_1\mathbf{m}_2) + \mathbf{\Omega}_{o_1} \times (A_2\mathbf{m}_1) + \mathbf{\Omega}_{o_2} \times (A_1\mathbf{m}_2) + \boldsymbol{\omega}_1 \times (A_2\mathbf{M}_{o_1})$$
$$+ \boldsymbol{\omega}_2 \times (A_1\mathbf{M}_{o_2}) + \mathbf{t}_1 \times (A_2\mathbf{n}_1) + \mathbf{t}_2 \times (A_1\mathbf{n}_2) + \mathbf{g} = \mathbf{0}. \tag{30}$$

5.2. *Kinematical relations*

We may use $\boldsymbol{\varepsilon}_\alpha$ and $\boldsymbol{\omega}_\alpha$ as the measures of deformation—$\boldsymbol{\varepsilon}_\alpha$ being the stretching-shearing strain vector and $\boldsymbol{\omega}_\alpha$ denoting the change of the curvature–twist vector. Following an inductive reasoning, or utilizing the principle of virtual work, we may express the strain–displacement and the curvature–rotation relationships as

$$\boldsymbol{\varepsilon}_\alpha = \frac{1}{A_\alpha}\frac{\partial \mathbf{u}}{\partial \theta_\alpha} + \mathbf{t}_\alpha \times \boldsymbol{\phi}$$
$$= \frac{1}{A_\alpha}[(\mathbf{u}_{,\alpha} + \mathbf{\Omega}_{o_\alpha} \times \mathbf{u}) + \mathbf{t}_\alpha \times \boldsymbol{\phi}] \quad (\alpha = 1, 2) \tag{31}$$

and

$$\omega_\alpha = \frac{1}{A_\alpha}\frac{\partial \phi}{\partial \theta_\alpha} = \frac{1}{A_\alpha}(\phi_{,\alpha} + \Omega_{o_\alpha} \times \phi). \tag{32}$$

A more rigorous derivation of these strain measures can be found in Whitman and Cohn (1978).

5.3. *Constitutive relations*

Using a linear elastic model for the shell material, we may express the constitutive equations as

$$\mathbf{n}_\alpha = \mathbf{J}_\alpha \varepsilon_\alpha \tag{33}$$

$$(a = 1, 2)$$

$$\mathbf{m}_\alpha = \mathbf{I}_\alpha \omega_\alpha \tag{34}$$

where $\mathbf{J}_\alpha$ and $\mathbf{I}_\alpha$ denote the membranal and flexural rigidities of the shell and are defined by the following dyadics:

$$\mathbf{J}_\alpha = J_{i\alpha}\mathbf{e}_i\mathbf{e}_i \qquad \begin{pmatrix} \alpha = 1, 2 \\ i = 1, 2, 3 \end{pmatrix}$$

$$\mathbf{I}_\alpha = I_{i\alpha}\mathbf{e}_i\mathbf{e}_i. \tag{35}$$

The totality of eqns (29)–(35) constitutes the compact linear stability equations of general shells subjected to motion-dependent loading. In these equations, the effects of prebuckling deformation, and local rotations are present. We also note that in these equations the critical state is not necessarily a membrane state and may also contain a flexural field. Further, account is taken of shear deformations in the sense of Reissner's transverse shear deformation. That is, in view of the zero stretch in the thickness coordinate, shear deformations are somewhat constrained.

6. THE EXPANDED, COMPONENT FORM OF GOVERNING EQUATIONS

In this section, and in the sequel, we shall assume that the prebuckling state is a membrane state, i.e. $\mathbf{M}_{o_\alpha} \equiv 0$. We shall now write down the expanded form of the governing vectorial equations, eqns (29)–(34). To arrive at more special, classical theories we shall also impose further constraints on these general equations.

6.1. *Equilibrium*

$$(A_2 n_{11})_{,1} + (A_1 n_{21})_{,2} + A_2(-an_{13} - cn_{12}) + A_1(on_{23} + dn_{12}) + A_2(\omega_{12}N_{o_{11}} - \omega_{13}N_{o_{12}}) + A_1(\omega_{22}N_{o_{23}} - \omega_{23}N_{o_{22}}) = 0$$

$$(A_2 n_{12})_{,1} + A_2(cn_{11}) + (A_1 n_{22})_{,2} + A_1(-dn_{21} - bn_{23}) + A_2(\omega_{13}N_{o_{11}} - \omega_{11}N_{o_{13}}) + A_1(\omega_{23}N_{o_{21}} - \omega_{21}N_{o_{23}}) = 0$$

$$(A_2 n_{13})_{,1} + A_2(-an_{11}) + (A_1 n_{23})_{,2} + A_1(bn_{22}) + A_2(\omega_{11}N_{o_{12}} - \omega_{12}N_{o_{11}}) + A_1(\omega_{21}N_{o_{22}} - \omega_{22}N_{o_{21}}) = 0$$

$$(A_2 m_{11})_{,1} + A_2(-am_{13} - cm_{12}) + (A_1 m_{21})_{,2} + A_1(dm_{22}) + A_1 A_2 n_{23} = 0$$

$$(A_2 m_{12})_{,1} + A_2(cm_{11}) + (A_1 m_{22})_{,2} A_1(-dm_{21} - bm_{23}) - A_1 A_2 n_{13} = 0$$

$$(A_2 m_{13})_{,1} + A_2(am_{11}) + (A_1 m_{23})_{,2} + A_1(bm_{22}) + A_1 A_2 n_{12} - A_1 A_2 n_{21} = 0. \tag{36}$$

6.2. *Kinematic relations*

Noting that

$$\boldsymbol{\varepsilon}_\alpha = \varepsilon_{\alpha j}\mathbf{e}_j = \frac{1}{A_\alpha}(\mathbf{u}_{,\alpha}+\boldsymbol{\Omega}_{o_\alpha}\times\mathbf{u})+\mathbf{t}_\alpha\times\boldsymbol{\phi}$$

and expressing the vectors $\boldsymbol{\Omega}_{o_\alpha}$, ($\alpha = 1,2$) in terms of their components o, $-a$, c, b, $-d$, as noted in eqns (6) and (7), we find

$$\varepsilon_{11} = \frac{1}{A_1}[u_{1,1}+(-au_3-cu_2)]$$

$$\varepsilon_{12} = \frac{1}{A_1}[u_{2,1}+(cu_1-ou_3)]-\phi_3$$

$$\varepsilon_{13} = \frac{1}{A_1}[u_{3,1}+(ou_2+au_1)]+\phi_2. \tag{37}$$

Similarly

$$\varepsilon_{21} = \frac{1}{A_2}[u_{1,2}+(ou_3+du_2)]+\phi_3$$

$$\varepsilon_{22} = \frac{1}{A_2}[u_{2,2}+(-du_1-bu_3)]$$

$$\varepsilon_{23} = \frac{1}{A_2}[u_{3,2}+(bu_2-ou_1)]-\phi_1. \tag{38}$$

If the shearing strains normal to the shell surface are to be neglected then we need to introduce the constraint relations

$$\varepsilon_{13} = 0 \quad \therefore \phi_2 = -\frac{1}{A_1}[u_{3,1}+(ou_2+au_1)]$$

$$\varepsilon_{23} = 0 \quad \therefore \phi_1 = \frac{1}{A_2}[u_{3,2}+(bu_2-ou_2)]. \tag{39}$$

For the flexural–torsional part of deformation, the curvature changes may be expressed as

$$\omega_\alpha = \omega_{\alpha j}\mathbf{e}_j = \frac{1}{A_\alpha}(\boldsymbol{\phi}_{,\alpha}+\boldsymbol{\Omega}_{o_\alpha}\times\boldsymbol{\phi}) \quad \alpha = 1,2$$

$$\omega_\alpha = \omega_{\alpha j}\mathbf{e}_j = \frac{1}{A_\alpha}(\boldsymbol{\phi}_{,\alpha}+\boldsymbol{\Omega}_{o_\alpha}\times\boldsymbol{\phi}) \quad \alpha = 1,2$$

or

$$\omega_{11} = \frac{1}{A_1}[\phi_{1,1}+(-a\phi_3-c\phi_2)]$$

$$\omega_{12} = \frac{1}{A_1}[\phi_{2,1}+(c\phi_1-o\phi_3)]$$

$$\omega_{13} = \frac{1}{A_1}[\phi_{3,1}+(o\phi_2+a\phi_1)]. \tag{40}$$

Similarly

$$\omega_{21} = \frac{1}{A_2}[\phi_{1,2} + (-o\phi_3 + d\phi_2)]$$

$$\omega_{22} = \frac{1}{A_2}[\phi_{2,2} + (-d\phi_1 - b\phi_3)]$$

$$\omega_{23} = \frac{1}{A_2}[\phi_{3,2} + (b\phi_2 - o\phi_1)]. \tag{41}$$

7. GENERAL STABILITY EQUATIONS OF CYLINDRICAL SHELLS

In this section we derive the axisymmetric stability equations of circular cylindrical shells, with local rotations, from the general equations derived in the previous sections. In this case we have that (see p. 319 of Donnel (1976))

$$\Omega_{o_{11}} = 0, \quad \Omega_{o_{12}} = -a = o, \quad \Omega_{o_{13}} = c = 0$$

$$\Omega_{o_{21}} = b = 1, \quad \Omega_{o_{22}} = 0, \quad \Omega_{o_{23}} = -d = 0$$

$A_1 = 1$, $A_2 = R$ (R is the radius of the circular profile)

$$\mathbf{t}_{o_1} = (1, 0, 0), \quad \mathbf{t}_{o_2} = (0, R, 0).$$

Using this data in eqns (36), and noting that due to zero shear prebuckling membranal state $N_{o_{12}} = N_{o_{21}} = 0$, we find the equilibrium equations as

$$(Rn_{11})_{,1} + (n_{21})_{,2} + R(-\omega_{13}N_{o_{12}}) + (-\omega_{23}N_{o_{22}}) = 0$$

$$(Rn_{12})_{,1} + (n_{22})_{,2} - n_{23} + R(\omega_{13}N_{o_{11}}) + \omega_{23}N_{o_{21}} = 0$$

$$(Rn_{13})_{,1} + (n_{23})_{,2} + n_{22} + R(\omega_{11}N_{o_{12}} - \omega_{12}N_{o_{11}}) + (\omega_{21}N_{o_{22}} - \omega_{22}N_{o_{21}}) = 0$$

$$(Rm_{11})_{,1} + (m_{21})_{,2} + Rn_{23} = 0$$

$$(Rm_{12})_{,1} + (m_{22})_{,2} - m_{23} - Rn_{13} = 0$$

$$(Rm_{13})_{,1} + (m_{23})_{,2} + m_{22} + Rn_{12} - Rn_{21} = 0. \tag{42}$$

The above equations are the stability equations of circular cylindrical shells in their general form embodying local rotations. These equations contain the classical stability equations which have been used in the literature (see, e.g. Farshad (1977) as a special case). To cast these equations into more familiar form, we introduce a more standard definition for the force quantities. If, as usual, x and θ represent the longitudinal and hoop coordinates of the cylinder, then the translation into familiar notation proceeds as follows:

$$m_{12} = m_{xx}, \quad m_{22} = m_{\theta x}$$

$$m_{11} = -m_{x\theta}, \quad m_{21} = -m_{\theta\theta}$$

$$m_{13} = m_{xz}, \quad m_{23} = -m_{\theta z}$$

and

$$n_{11} = n_{xx}, \quad n_{21} = n_{\theta x}, \quad N_{o_{11}} = N_{o_{xx}}$$

$$n_{12} = n_{x\theta}, \quad n_{22} = n_{\theta\theta}, \quad N_{o_{22}} = N_{o_{\theta\theta}}$$

$$n_{13} = n_{xz}, \quad n_{23} = n_{\theta z}, \quad N_{o_{12}} = N_{o_{x\theta}}, \quad N_{o_{21}} = N_{o_{\theta x}};$$

introducing this notation into eqns (42) obtain

$$
\begin{aligned}
Rn_{xx,x}+n_{\theta x,\theta}-R\omega_{13}N_{o_{xx}}-\omega_{23}N_{o_{\theta\theta}} &= 0\\
Rn_{x\theta,x}+n_{\theta\theta,\theta}+-n_{\theta z}R\omega_{13}N_{o_{xx}}+\omega_{23}N_{o_{\theta x}} &= 0\\
Rn_{xz,x}+n_{\theta z,\theta}+n_{\theta\theta}+R(\omega_{11}N_{o_{x\theta}}-\omega_{12}N_{o_{xx}})+(\omega_{21}N_{o_{\theta\theta}}-\omega_{22}N_{o_{\theta x}}) &= 0\\
-Rm_{x\theta,x}-m_{\theta\theta,\theta}+Rn_{\theta z} &= 0\\
Rm_{xx,x}+m_{\theta x,\theta}+m_{\theta z}-Rn_{xz} &= 0\\
Rm_{xz,x}+m_{\theta z,\theta}+m_{\theta\theta}+Rn_{x\theta}-Rn_{\theta x} &= 0.
\end{aligned}
\tag{43}
$$

Now, to obtain the special (classical) stability equations of cylindrical shells we suppress the local rotation ϕ_3 and hence we set $\omega_{13}=\omega_{23}\equiv 0$. In that way, part of the moment–curvature relation (34) would result in $m_{xz}=m_{\theta z}=0$ and, therefore, for identical satisfaction of the last equation we deduce that $N_{x\theta}\equiv N_{\theta x}$. Upon this specialization, eqns (43) yield

$$
\begin{aligned}
Rn_{xx,x}+n_{\theta x,\theta} &= 0\\
Rn_{x\theta,x}+n_{\theta\theta,\theta}-n_{\theta z} &= 0\\
Rn_{xz,x}+n_{\theta z,\theta}n_{\theta\theta}+R(\omega_{11}N_{o_{x\theta}}-\omega_{12}N_{o_{xx}})+(\omega_{21}N_{o_{\theta\theta}}-\omega_{22}N_{o_{\theta x}})-Rm_{x\theta,x}-m_{\theta\theta,\theta}+Rn_{\theta z} &= 0\\
Rm_{xx,x}+m_{\theta x,\theta}-m_{\theta z}-Rn_{xz} &= 0.
\end{aligned}
\tag{44}
$$

The more familiar notation for the rate of change of curvature and twist components ($\omega_{11},\omega_{12},\omega_{21},\omega_{22}$) can also be written from kinematic relations (39)–(41). Thus, with the help of the above-mentioned notational translations, suppressing ϕ_3 and further assuming that

$$u_1=u,\quad u_2=v,\quad u_3=w$$

we may write

$$
\begin{aligned}
\omega_{11}&=w_{,xx}, & \omega_{12}&=-w_{,xx}\\
\omega_{21}&=w_{,\theta\theta}, & \omega_{22}&=w_{,x\theta}.
\end{aligned}
\tag{45}
$$

Now, by combining eqns $(44)_2$–$(44)_4$ and making use of notation (45) we finally arrive at the following set of equations:

$$
\begin{aligned}
Rn_{xx,x}+n_{\theta x,\theta} &= 0\\
Rn_{x\theta,x}+n_{\theta\theta,\theta}-n_{\theta z} &= 0\\
m_{xx,x}+\frac{2}{R}m_{x\theta,x\theta}+\frac{1}{R^2}m_{\theta\theta,\theta\theta}+n_{\theta\theta}+w_{,xx}N_{o_{xx}}+2w_{,x\theta}N_{o_{x\theta}}+w_{,\theta\theta}N_{o_{\theta\theta}} &= 0
\end{aligned}
\tag{46}
$$

which are the well-known stability equations of cylindrical shells as derived and utilized in the literature (Farshad, 1977; Farshad and Ahmadi, 1979; Tabarrok and Dost, 1978).

8. A CONTINUUM MODEL FOR STABILITY OF GRIDWORKS

The stability equations established in this paper can be used to derive the in-plane and out-of-plane stability equations of flat plates with local rotations taken into consideration. In this sense, they would then also constitute a continuum model for the stability analysis of planar gridworks in which the effect of local (joint) rotations must always be taken into consideration.

To arrive at the stability equations of such generalized plates we let $a = b = c = d = 0$ and set $A_1 = A_2 = 1$. The governing equations, eqns (36)–(41), then yield

$$\begin{aligned} n_{11,1}+n_{21,2}-\omega_{13}N_{o_{12}}-\omega_{23}N_{o_{22}} &= 0 \\ n_{12,1}+n_{22,2}+\omega_{13}N_{o_{11}}+\omega_{23}N_{o_{21}} &= 0 \\ n_{13,1}+n_{23,2}+(\omega_{11}N_{o_{12}}-\omega_{12}N_{o_{11}})+(\omega_{21}N_{o_{22}}-\omega_{22}N_{o_{21}}) &= 0 \\ m_{11,1}+m_{21,2}+n_{23} &= 0 \\ m_{12,1}+m_{22,2}-n_{13} &= 0 \\ m_{13,1}+m_{23,2}+n_{12}-n_{21} &= 0 \end{aligned} \tag{47}$$

and

$$\begin{aligned} \phi_1 &= u_{3,2}, & \phi_2 &= -u_{3,1} \\ \omega_{11} &= u_{3,12}, & \omega_{21} &= u_{3,22} \\ \omega_{12} &= -u_{3,11}, & \omega_{22} &= -u_{3,12} \\ \omega_{13} &= \phi_{3,1}, & \omega_{23} &= \phi_{3,2} \end{aligned} \tag{48}$$

for a generally orthotropic plate and/or gridwork, constitutive relations (33) and (34) with the help of relations (37)–(41), may be written as

$$\begin{aligned} n_{11} &= J_1u_{1,1}+J_2u_{2,2} \\ n_{22} &= J_2u_{1,1}+J_1u_{2,2} \\ n_{12} &= J_3(u_{2,1}-\phi_3) \\ n_{21} &= J_4(u_{1,2}+\phi_3) \end{aligned} \tag{49}$$

$$\begin{aligned} m_{11} &= I_1\omega_{11} \\ m_{22} &= I_2\omega_{22} \\ m_{12} &= I_3\omega_{12}+I_4\omega_{21} \\ m_{21} &= I_4\omega_{12}+I_3\omega_{21} \\ m_{13} &= I_5\phi_{3,1} \\ m_{23} &= I_5\phi_{3,2}. \end{aligned} \tag{50}$$

The general plate stability equations, eqns (47)–(50), embody, as a special case, the so-called "Cosserat plate" equations (Farshad and Tabarrok, 1979) and also the classical stability equations of plates. These equations can be utilized in the stability analyses of plates, gridworks and fiber-reinforced composites undergoing in-plane as well as out-of-plane modes of elastic instability. As an application, a specific example of in-plane stability analysis of a gridwork and/or fiber-reinforced composite plate is worked out in the following section.

9. AN EXAMPLE

Let $u_3 \equiv 0$ and neglect Poisson-type effects in relations (49) and (50), so that $J_2 = 0$, $I_4 = 0$. Also let $\alpha = I_3/I_1$ and $\beta = I_5/I_1$. Consider, for instance, the case of uni-directional compression loading $N = -N_{o_{11}}/I_1$, $N_{o_{12}} = N_{o_{21}} = 0$, $N_{o_{22}} = 0$. Under these assumptions, the combination of eqns (47)–(50) yields

$$\begin{aligned} u_{1,11}+\alpha(u_{1,22}+\phi_{3,2}) &= 0 \\ \alpha(u_{2,11}+\phi_{3,1})+u_{2,22}-\phi_{3,1}N &= 0 \\ \beta(\phi_{3,11}+\phi_{3,22})+\alpha(u_{2,1}-u_{1,2}-2\phi_3) &= 0. \end{aligned} \tag{51}$$

We consider the symmetric–symmetric mode of in-plane buckling of a simply supported square gridwork (of sides $2a$) represented by the following appropriate solution:

$$\begin{aligned} u_1 &= A_1 \sin\frac{\pi x_1}{a}\cos\frac{\pi x_2}{a} \\ u_2 &= A_2 \cos\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a} \\ \phi_3 &= A_3 \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}. \end{aligned} \tag{52}$$

Substituting this solution into eqns (51) and combining them we obtain the following characteristic equation:

$$-\frac{\pi^2}{a_2}\beta-2\alpha+\frac{\alpha^2}{1+\alpha}+\frac{\alpha(\alpha+N)}{1+\alpha}=0$$

from which we find the critical load for symmetric–symmetric in-plane buckling as

$$N_{cr} = \frac{1+\alpha}{\alpha}\left(\frac{\pi^2}{a^2}\beta+2\alpha\right).$$

10. CONCLUDING REMARKS

The presently derived stability equations of general shells are, in our opinion, compact, mathematically convenient and, physically plausible. In addition to embodying the behavioral features found in the classical shell theories, which are derivable as their special cases, these equations contain the influence of shearing deformation and also the local rotations. The latter feature allows one to apply these equations in the stability analysis of composite shells and plates as well as gridworks in which the in-plane, membranal mode of instability becomes a matter of investigation. The general, vectorial equations derived herein are also suitable for the formulation of a general finite element model of shells. A similar approach used by the authors has been successfully applied for the analysis of spatially curved and pretwisted rods. As illustrated the present approach also facilitates the development of continuum models for the stability of flat and curved gridworks.

REFERENCES

Donnel, L. H. (1976). *Beams, Plates and Shells*. McGraw-Hill, New York.
Farshad, M. (1977). Membrane model of surface effects. *J. Engng Mech. Div., Proc. ASCE* **103**, 939–948.
Farshad, M. and Ahmadi, G. (1979). Influence of loading behaviour on the stability of cylindrical shells. *J. Sound Vibr.* **62**(4), 533–540.
Farshad, M. and Tabarrok, B. (1976). Dualities in the analysis of Cosserat plate. *Mech. Res. Commun.* **3**, 399–406.

Farshad, M. and Tabarrok, B. (1987). On stability equations of spatial rods—a technical theory. *Trans. CSME* **11**, 229–235.

Goldenveiser, A. L. (1961). *Theory of Elastic Thin Shells*. Pergamon Press, Oxford.

Green, A. E. and Naghdi, P. M. (1971). On superposed small deformations on a large deformation of an elastic Cosserat surface. *J. Elasticity* **1**, 1–17.

Green, A. E. and Zerna, W. (1968). *Theoretical Elasticity*, 2nd Edn. Oxford University Press, Oxford.

Koiter, W. T. (Editor) (1960). The theory of the elastic shells. *Proc. IUTAM Symp.* North-Holland, Amsterdam.

Love, E. H. (1944). *Mathematical Theory of Elasticity*, 4th Edn, p. 517. Dover Publications, New York.

Naghdi, P. M. (1972). The theory of plates and shells. In *Handbuch der Physik*, Vol. VIa/2. *Mechanics of Solids II*. (Edited by C. Truesdell). Springer, Berlin.

Niordson, F. I. (Editor) (1969). Theory of thin shells. *Proc. 2nd IUTAM Symp.* Springer, Berlin.

Tabarrok, B. and Dost, S. (1978). Application of stress functions to initial stress problems. *J. Appl. Mech.* **45**(2), 350–354.

Whitman, A. and Cohn, H. (1978). Constitutive equations for curved and twisted initially elastic rods. *Acta Mech.* **30**, 237.